PHYS-285 Hwk 2

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1. Recall that in the rest frame,

$$p = \langle E/c, 0, 0, 0 \rangle = \langle mc, 0, 0, 0 \rangle$$

So that

$$p^\mu p_\mu = m^2 c^2$$

Next, we consider a Lorentz boost of velocity v in the x direction; our Lorentz transformations are

$$p'_x = \gamma(p_x - vE/c^2) = -\frac{vm}{\sqrt{1 - v^2/c^2}}$$

$$E' = \gamma(E - vp_x) = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

with p_y, p_z unchanged at 0. Our new momentum is

$$p = \langle E'/c, p'_x, 0, 0 \rangle = \left\langle \frac{mc}{\sqrt{1 - v^2/c^2}}, -\frac{vm}{\sqrt{1 - v^2/c^2}}, 0, 0 \right\rangle$$

$$\begin{aligned} p^\mu p_\mu &= \left(\frac{mc}{\sqrt{1 - v^2/c^2}} \right)^2 - \left(\frac{vm}{\sqrt{1 - v^2/c^2}} \right)^2 \\ &= \frac{m^2c^2 - m^2v^2}{1 - v^2/c^2} \\ &= m^2 c^2 \end{aligned}$$

where the last division is easiest to see by writing the corresponding multiplication

$$m^2c^2(1 - v^2/c^2) = m^2c^2 - m^2v^2$$

2. In a frame S, a firecracker goes off at (50s, 150 000m, 12 000m, 3 000m).

   (a) In a frame S', boosted from S in the x direction at $-0.8c$, we use the Lorentz transformation

$$\begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 50c \\ 150000 \\ 12000 \\ 3000 \end{bmatrix} = \begin{bmatrix} 50c\gamma - 150000\beta\gamma \\ -50c\beta\gamma + 150000\gamma \\ 12000 \\ 3000 \end{bmatrix}$$

With $\gamma = (1 - 0.8^2)^{-1/2} = 5/3$, $\beta = -0.8$, $c = 3 \times 10^8$, this is

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 2.5 \times 10^{10} - 2 \times 10^5 \\ -2 \times 10^{10} + 2.5 \times 10^5 \\ 12000 \\ 3000 \end{bmatrix}$$

i.e.,

$$t' = 83.33\text{s} \qquad x' = -2 \times 10^8\text{km} \qquad y' = 12\text{km} \qquad z' = 3\text{km}$$

(b) Given the time light takes to travel, in the rest frame, the firecracker will be seen at

$$50\text{s} + \sqrt{x^2 + y^2 + z^2}/c = 50.001\text{s}$$

However, in the moving frame, this is

$$50\text{s} + \sqrt{x'^2 + y'^2 + z'^2}/c = 116.67\text{s}$$

3. If a meterstick makes an angle of $30°$, then its height (along the y axis) is 0.5m and length (along the x axis) is $\sqrt{3}/2$m.

   In the non-primed frame, the height will be the same, 0.5m, since there is no change in velocity along the y axis. So, if the meterstick makes an angle of $45°$, its *length* must contract to 0.5m to match the height.

   In equations,

$$\Delta x' = \frac{\sqrt{3}}{2} = \gamma \frac{1}{2} = \gamma \Delta x$$

   Solving for γ,

$$\frac{1}{\sqrt{1 - v^2/c^2}} = \gamma = \sqrt{3} \implies v = \sqrt{2/3}c$$

4. We seek

$$\max_v \left\{ \frac{\frac{1}{2}mv^2}{(\gamma - 1)mc^2} \geq 0.99 \right\}$$

   To simplify our lives, scale v and c so that $c = 1$. Then the v we seek satisfies

$$0.99 = \frac{v^2}{2\gamma - 2} = \frac{v^2}{\frac{2}{\sqrt{1-v^2}} - 2} = \frac{v^2\sqrt{1 - v^2}}{2 - 2\sqrt{1 - v^2}}$$

   After rearranging,

$$0 = -v^6 - 2.96v^4 + 0.0396v^2$$

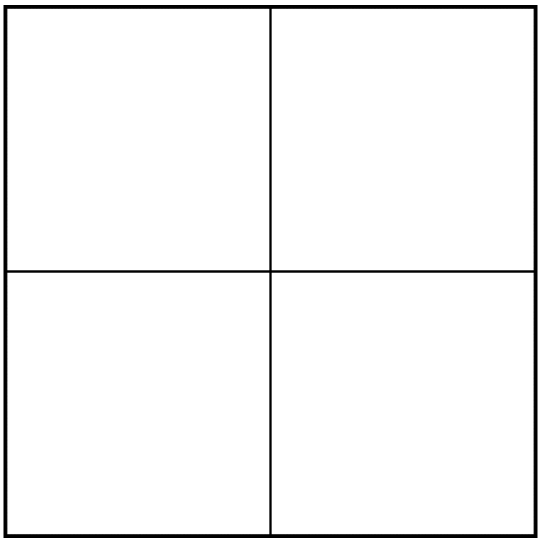
   Plugging this into sage, we get

$$v \approx 0.11540559911376866c = 3.46 \times 10^7\text{m/s}$$

5. Consider the collision as seen in the zero-momentum frame. Before the collision, momentum is zero and energy is positive. However, after the collision, the momentum must still be zero, but if there is nothing but a photon left, we will have

$$E = pc = 0c = 0$$

violating conservation of energy.



6. In the rest frame, we see the criminal at speed v_0, the cop at speed v_p, and the bullet at speed $\frac{v_p+v_m}{1+v_p v_m/c^2}$. Clearly, in this frame the bullet will hit the criminal iff

$$v_0 < v_{\text{bullet}} = \frac{v_p + v_m}{1 + v_p v_m/c^2}$$

Now, consider a Lorentz boost in the x direction of speed v. Then we have

$$v_0' = \frac{v_0 - v}{1 - \frac{v v_0}{c^2}}$$

$$v_{\text{bullet}}' = \frac{v_{\text{bullet}} - v}{1 - \frac{v v_{\text{bullet}}}{c^2}}$$

and again, the bullet will hit the criminal iff

$$v_0' < v_{\text{bullet}}'$$

$$\frac{v_0 - v}{1 - \frac{v v_0}{c^2}} < \frac{v_{\text{bullet}} - v}{1 - \frac{v v_{\text{bullet}}}{c^2}}$$

$$(v_0 - v)\left(1 - \frac{v v_{\text{bullet}}}{c^2}\right) < (v_{\text{bullet}} - v)\left(1 - \frac{v v_0}{c^2}\right)$$

$$v_0 - v - \frac{v v_0 v_{\text{bullet}}}{c^2} + \frac{v^2 v_{\text{bullet}}}{c^2} < v_{\text{bullet}} - v - \frac{v v_0 v_{\text{bullet}}}{c^2} + \frac{v^2 v_0}{c^2}$$

$$c^2 v_0 + v^2 v_{\text{bullet}} < c^2 v_{\text{bullet}} + v^2 v_0$$

$$c^2 (v_0 - v_{\text{bullet}}) < v^2 (v_0 - v_{\text{bullet}})$$

Now, since $v < c$ for any valid Lorentz transformation, this last inequality will hold iff

$$v_0 - v_{\text{bullet}} < 0$$

i.e.,

$$v_0 < v_{\text{bullet}}$$

So that the criminal outruns the bullet in the rest frame, iff he outruns the bullet in every other frame.

7. If this is a head-on collision, it seems like the angle between the final velocities should be 0, no?